THE PLEASURES OF COUNTING

Suppose there is a Petanque *triples* tournament where *3n* players are grouped and allocated to three hats *1, 2* and *3,* and n teams are selected by drawing, at random, a player from each hat.

What is the probability that player X in hat *1* will be in the same team as player Y in hat *3*?

It turns out that this probability is: $\Pr(X \text{ and } Y \text{ in same team}) = \frac{1}{n}$

How is this relatively simple formula obtained?

Consider the simple case where there are 12 players to be selected in $n = 4$ teams, and players *A,B,C,D* are in hat *1;* players *E,F,G,H* are in hat *2;* and players *I,J,K,L* are in hat *3.*

Here is a possible draw. First, the letters *D, B, A* and *C* are drawn from hat *1* and written down in order in column 2. Then the letters *G, F, H* and *E* are drawn from hat *2* and written down in order in column 3. Finally, the letters *L, I, K* and *J* are drawn from hat *3* and written down in order in column 4.

The teams are: (1) *D,G,L;* (2) *B,F,I;* (3) *A,H,K;* and *C,E,J.*

column 1	column 2	column 3	column 4
Team	hat *1*	hat *2*	hat *3*
1	*D*	*G*	*L*
2	*B*	*F*	*I*
3	***A***	*H*	***K***
4	*C*	*E*	*J*

What is the probability that players *A* and *K* will be in the same team? As they are above.

Now there are $n! = 4! = 4 \times 3 \times 2 \times 1 = 24$ possible arrangements of the letters *A,B,C,D* in hat *1,* where $n!$ means n-factorial. Similarly for the letters *E,F,G,H* in hat *2* and *I,J,K,L* in hat *3.* So the total number of possible teams is $n! \times n! \times n! = 24 \times 24 \times 24 = 13824$ since each of the $n!$ arrangements of *A,B,C,D* can be coupled with $n!$ arrangements of *E,F,G,H;* and these coupled again with the $n!$ arrangements of *I,J,K,L.*

Now; back to hat *1.* Amongst the $n! = 4! = 24$ arrangements of *A,B,C,D* there are $(n-1)! = 3! = 3 \times 2 \times 1 = 6$ arrangements where *A* is the first letter. These are: *ABCD, ACDB, ADCB, ADBC, ACBD* and *ABDC.* Also, there are $(n-1)!$ arrangements where *A* is the second letter, $(n-1)!$ arrangements where *A* is the third letter and $(n-1)!$ arrangements where *A* is the fourth letter. Similar reasoning applies to arrangements of individual letters in hats *2* and *3.*

After drawing from hats *1, 2* and *3* the possibilities where *A* and *K* are in team 1 are:

A,E,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,F,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,G,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,H,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,.$

The possibilities where *A* and *K* are in team 2 are:

A,E,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,F,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,G,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,;$

A,H,K: $(n-1)! \times (n-1)! \times (n-1)! = 6 \times 6 \times 6 = 216\,.$

And similarly for teams 3 and 4.

So the number of possible arrangements where *A* and *K* are together in the one team are:

$$n \times n \times (n-1)! \times (n-1)! \times (n-1)! = 4 \times 4 \times 6 \times 6 \times 6 = 3456$$

The probability that A and K are together in the same team is:

$$\Pr(A \text{ and } K \text{ together}) = \frac{\text{number of possibilities}}{\text{total number of arrangements}} = \frac{3456}{13824} = \frac{1}{4}$$

This could also be expressed as

$$\begin{aligned}\Pr(A \text{ and } K \text{ together}) &= \frac{\text{number of possibilities}}{\text{total number of arrangements}} \\ &= \frac{n \times n \times (n-1)! \times (n-1)! \times (n-1)!}{n! \times n! \times n!}\end{aligned}$$

Noting that $n! = n \times (n-1)!$ this result for A and K can be generalised as the probability of player X (from hat *1*) being in the same team as player Y (from hat *2*) as

$$\Pr(X \text{ and } Y \text{ together}) = \frac{(n-1)!}{n!} = \frac{(n-1)!}{n \times (n-1)!} = \frac{1}{n}$$

For a triples tournament where there are $3n = 54$ players arranged in three hats containing $n = 18$ players, and player X is in hat *1* and player Y is in hat *2;* the probability that X and Y are in the same team after a random selection (one player from each hat) is $\frac{1}{18} = 0.05555\ldots$

Rod Deakin,

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